

A regularization of field theory on non-commutative torus

NAOFUMI KITSUNEZAKI* and SHOZO UEHARA†

*Department of Physics, Nagoya University,
 Chikusa-ku, Nagoya 464-8602, Japan*

Abstract

Matrix model is used as a regularization of field theory on non-commutative torus. However, there exists an example that the product of the large- N limit of matrices does not coincide with that of the corresponding fields. We propose a new procedure for regularizing fields on a non-commutative torus by matrices with the help of the projection in the representation space, so that the products of the matrices coincide with those of the corresponding fields in the large- N limit.

1 Introduction

It is known that the low energy effective theory with constant background $B_{\mu\nu}$ fields on a D-brane is a field theory in the non-commutative spacetime [1]. And field theory in the non-commutative spacetime, which we call non-commutative field theory (NCFT), has been investigated in view of studying properties of superstring theories.

It is reported that some open Wilson lines are gauge invariant in non-commutative space-time in Yang-Mills theory [2]. The stringy properties of NCFT were discovered in [3] using perturbative approach, which is very similar to matrix theories. Indeed, it was shown that real fields on non-commutative discrete periodic lattice (NCDPL) of $N \times N$ sites can be mapped to $N \times N$ Hermitian matrices, and their integration over the lattice are mapped to the trace of the corresponding matrices [4]. To construct the basis of fields over NCDPL of $N \times N$ sites, let us consider

$$T_{(m,n)}^{(N)}(\hat{x}, \hat{y}) = e^{2i\kappa(m\hat{x}+n\hat{y})}, \quad (1)$$

where κ is a constant which will be determined later, $[\hat{x}, \hat{y}] = i$ and m and n are integers, which makes the (\hat{x}, \hat{y}) -space periodic,

$$T_{(m,n)}^{(N)}(\hat{x} + k\alpha, \hat{y} + l\alpha) = T_{(m,n)}^{(N)}(\hat{x}, \hat{y}). \quad (k, l \in \mathbb{Z}, \alpha = \pi/\kappa) \quad (2)$$

In order to regularize by matrix we will impose periodicity on m and n to restrict the representation space of eq.(1) by the condition

$$e^{2i\kappa N\hat{x}} = e^{2i\kappa N\hat{y}} = 1. \quad (3)$$

*Email address: kitsune@eken.phys.nagoya-u.ac.jp

†Email address: uehara@eken.phys.nagoya-u.ac.jp

Actually this leads to $T_{(m+kN, n+lN)}^{(N)}(\hat{x}, \hat{y}) = T_{(m, n)}^{(N)}(\hat{x}, \hat{y})$ and also this means that the (\hat{x}, \hat{y}) -space is now discrete. Furthermore, κ is set $\sqrt{\pi M/N}$ where M and N are mutually prime since $e^{2i\kappa N \hat{x}}$ should commute with any $T_{(m, n)}^{(N)}(\hat{x}, \hat{y})$, so that the commutation relation becomes

$$[T_{(m_1, m_2)}^{(N)}(\hat{x}, \hat{y}), T_{(n_1, n_2)}^{(N)}(\hat{x}, \hat{y})] = -2i \sin \left(2\pi \frac{M}{N} (m_1 n_2 - m_2 n_1) \right) T_{(m_1 + n_1, m_2 + n_2)}^{(N)}(\hat{x}, \hat{y}). \quad (4)$$

Then, the matrix basis corresponding to eq.(1) are written by the clock and the shift matrices,

$$M_{(m, n)}^{(N)} = e^{i2\pi \frac{M}{N} mn} \begin{pmatrix} 1 & & & \\ & e^{4\pi i \frac{M}{N}} & & \\ & & \ddots & \\ & & & e^{4\pi i \frac{M}{N} (N-1)} \\ 0 & & & \end{pmatrix}^m \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 1 & 0 & & 0 \end{pmatrix}^n. \quad (5)$$

Due to this correspondence, the real fields on NCDPL are mapped to Hermitian matrices, the star products of fields are mapped to products of the corresponding matrices and the integration over NCDPL coincide with the trace of the corresponding matrix [4]. These facts suggest that NCFT on torus can be regularized by the corresponding matrix model.

On the other hand, we find that the matrices in eq.(5) can be expressed by the following operators on the Fock space,

$$\mathcal{T}_{(m, n)}^{(N)} = \sum_{k=0}^{N-n-1} e^{i2\pi \frac{M}{N} (2km+mn)} |k\rangle \langle k+n| + \sum_{k=0}^{n-1} e^{i2\pi \frac{M}{N} (2km-mn)} |N-n-k\rangle \langle k|, \quad (6)$$

where

$$|k\rangle = \frac{1}{\sqrt{k!}} (\hat{a}^\dagger)^k |0\rangle, \quad (\hat{a}|0\rangle = 0) \quad (7)$$

and

$$\hat{a} = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{y}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{x} - i\hat{y}). \quad (8)$$

Since $T_{(m, n)}^{(N)}(\hat{x}, \hat{y})$ in eq.(1) and $\mathcal{T}_{(m, n)}^{(N)}$ in eq.(6) are isomorphic, we can expect that their large- N limits will also be isomorphic, so that NCFT on a torus will be completely reproduced by the large- N limit of the matrix models. However, there exists a counterexample: In the large- N limit, the operators $T_{(0, \pm 1)}^{(N)}$ have the following weak convergence limits,¹

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathcal{T}_{(0, 1)}^{(N)} (\equiv \mathcal{T}_{(0, 1)}^{(\infty)}) &= \sum_{k=0}^{\infty} |k\rangle \langle k+1|, \\ \lim_{N \rightarrow \infty} \mathcal{T}_{(0, -1)}^{(N)} (\equiv \mathcal{T}_{(0, -1)}^{(\infty)}) &= \sum_{k=0}^{\infty} |k+1\rangle \langle k|, \end{aligned} \quad (9)$$

and their product is calculated as

$$\mathcal{T}_{(0, -1)}^{(\infty)} \mathcal{T}_{(0, 1)}^{(\infty)} = 1 - |0\rangle \langle 0|. \quad (10)$$

¹We cannot discuss the convergence of the operators with the “norm” which is naturally defined by the trace of the operator, since such a “norm” is divergent for some operators.

On the other hand, the product of the corresponding plane waves in eq.(1) are

$$T_{(0,-1)}^{(\infty)}(\hat{x}, \hat{y}) T_{(0,1)}^{(\infty)}(\hat{x}, \hat{y}) = 1, \quad (11)$$

which obviously differs from eq.(10). This raises a question that the matrix models might not regularize NCFT.

In this paper, we consider the following projected plane-wave like (PPWL) operators,

$$\tilde{T}_{(m,n)}^{(N)} = \sum_{k,l=0}^{N-1} |k\rangle\langle k| e^{i2\pi\lambda(m\hat{a}+in\hat{a}^\dagger)} |l\rangle\langle l|, \quad (12)$$

where λ is an arbitrary real number and \hat{a}, \hat{a}^\dagger are given in eq.(8), and hence all the operators of $\exp[2\pi i\lambda(m\hat{a} + in\hat{a}^\dagger)]$ are equivalent to the plane waves $\exp[2\pi i\lambda(m\hat{x} + n\hat{y})]$.² And we show that N^2 numbers of PPWL operators made of the lowest N states in eq.(12) can be used as the bases of $N \times N$ Hermitian matrices instead of the clock and the shift matrices in eq.(6). The advantage of using PPWL operators are that the products of the projected plane waves in eq.(12) are equal to those of the original plane waves even in the large- N weak convergence limit,

$$\lim_{N \rightarrow \infty} \left| \sum_{p=0}^{N-1} \langle k | e^{i2\pi\lambda(m_1\hat{a}+im_2\hat{a}^\dagger)} | p \rangle \langle p | e^{i2\pi\lambda(n_1\hat{a}+in_2\hat{a}^\dagger)} | l \rangle - \langle k | e^{i2\pi\lambda(m_1\hat{a}+im_2\hat{a}^\dagger)} e^{i2\pi\lambda(n_1\hat{a}+in_2\hat{a}^\dagger)} | l \rangle \right| = 0. \quad (13)$$

In eq.(13), we have used the fact that the projection operator $P_N = \sum_{p=0}^{N-1} |p\rangle\langle p|$ becomes an identity operator in the large- N weak convergence limit, so that there is no such problem as the discrepancy between eqs.(10) and (11). Then the question to ask is whether N^2 numbers of PPWL operators are independent or not, which is shown in the next section. The final section is devoted to summary and discussion.

2 Independence of PPWL operators

In this section, we show that any N^2 numbers of the PPWL operators in eq.(12) are independent, i.e., $\sum_{m,n} a_{(m,n)} \tilde{T}_{(m,n)}^{(N)} = 0$ leads to $a_{(m,n)} = 0$ for all (m,n) . Let us define $N^2 \times N^2$ matrices $X^{(N)} = \left(X_{(m,n)(k,l)}^{(N)} \right)$ as

$$X_{(m,n)(k,l)}^{(N)} \equiv \langle k | e^{i2\pi\lambda(m\hat{a}+in\hat{a}^\dagger)} | l \rangle \quad (14)$$

$$= e^{-\pi^2\lambda^2(m^2+n^2)} \sqrt{k!l!} \sum_{p=0}^{\min\{k,l\}} \frac{(i\sqrt{2\pi}\lambda m)^{k-p} (-\sqrt{2\pi}\lambda n)^{l-p}}{p!(k-p)!(l-p)!}, \quad (15)$$

where (m,n) run over some different N^2 points in \mathbb{Z}^2 and $0 \leq k, l \leq N-1$. Then the above statement is equivalent to the one that the determinant of the $N^2 \times N^2$ matrix $X^{(N)}$

²They are equivalent but not *unitary* equivalent.

is non-zero. Since the factor $e^{-\pi^2 \lambda^2 (m^2 + n^2)}$ in eq.(15) depends only on the number of the rows of the matrix $X^{(N)}$ while $\sqrt{k!l!}$ on the number of the columns, we have

$$\det X^{(N)} = \left(\prod_{m,n} e^{-\pi^2 \lambda^2 (m^2 + n^2)} \right) \left(\prod_{k,l=0}^{N-1} \frac{1}{\sqrt{k!l!}} \right) \det \tilde{X}^{(N)}, \quad (16)$$

where

$$\tilde{X}_{(m,n)(k,l)}^{(N)} = k!l! \sum_{p=0}^{\min\{k,l\}} \frac{(i\sqrt{2}\pi\lambda m)^{k-p} (-\sqrt{2}\pi\lambda n)^{l-p}}{p! (k-p)! (l-p)!}, \quad (17)$$

$$= (i\sqrt{2}\pi\lambda m)^k (-\sqrt{2}\pi\lambda n)^l + kl (i\sqrt{2}\pi\lambda m)^{k-1} (-\sqrt{2}\pi\lambda n)^{l-1} + \frac{k(k-1)l(l-1)}{2} (i\sqrt{2}\pi\lambda m)^{k-2} (-\sqrt{2}\pi\lambda n)^{l-2} + \dots, \quad (18)$$

so that we have only to show $\det \tilde{X}^{(N)} \neq 0$. It is still difficult to directly calculate the determinant. Notice that the second term in eq.(18) is proportional to the first term of the corresponding expansion of $\tilde{X}_{(m,n)(k-1,l-1)}^{(N)}$, the third term in eq.(18) is proportional to the second term of the expansion of $\tilde{X}_{(m,n)(k-1,l-1)}^{(N)}$ which is also proportional to the first term of the expansion of $\tilde{X}_{(m,n)(k-2,l-2)}^{(N)}$ and so on. Then we can write

$$\tilde{X}_{(m,n)(k,l)}^{(N)} = (i\sqrt{2}\pi\lambda m)^k (-\sqrt{2}\pi\lambda n)^l + kl \tilde{X}_{(m,n)(k-1,l-1)}^{(N)} + \dots \quad (19)$$

It is obvious that the ellipsis in eq.(19) can be rewritten by a proper linear combination of the elements $\{\tilde{X}_{(m,n)(k-2,l-2)}^{(N)}, \dots, \tilde{X}_{(m,n)(k-t,l-t)}^{(N)}\}$ where $t = \min\{k, l\}$, i.e., we find that there exist such coefficients $\{a_p^{(k,l)}\}_{p=1}^{\min\{k,l\}}$ that the matrix element $\tilde{X}_{(m,n)(k,l)}^{(N)}$ can be written by

$$\tilde{X}_{(m,n)(k,l)}^{(N)} = (i\sqrt{2}\pi\lambda m)^k (-\sqrt{2}\pi\lambda n)^l + \sum_{p=1}^{\min\{k,l\}} a_p^{(k,l)} \tilde{X}_{(m,n)(k-p,l-p)}^{(N)}. \quad (20)$$

Thus, due to the fundamental property of the determinant,³ we have

$$\det \tilde{X}^{(N)} = \det Y^{(N)}, \quad (21)$$

where the matrix elements of $Y^{(N)} = (Y_{(m,n)(k,l)}^{(N)})$ is given by

$$Y_{(m,n)(k,l)}^{(N)} = (i\sqrt{2}\pi\lambda m)^k (-\sqrt{2}\pi\lambda n)^l, \quad (22)$$

which is given by the direct product of the Vandermonde matrices, and hence the determinant of $Y^{(N)}$ is non-zero. Thus we have shown that any N^2 numbers of PPWL operators in eq.(12) made of the lowest N states are independent. Especially, if we take the set of operators $\{\tilde{T}_{(m,n)}^{(N)} : -(N-1)/2 \leq m, n \leq (N-1)/2\}$, which are of course independent, we can easily take a proper linear combination to make Hermitian elements, so that we can use them as the bases of $N \times N$ Hermitian matrices.

³The coefficients $a_p^{(k,l)}$ do not depend on m and n .

3 Summary and discussion

We have studied how field theory on the non-commutative torus is regularized by $N \times N$ Hermitian matrix theory. In the usual regularization with the clock and the shift matrices in eq.(5), some products of the matrices do not recover the original products of the fields even in the large- N limit as we have seen in eqs.(10) and (11), which concerns that matrix theory could not reproduce field theory on non-commutative torus even in the large- N limit. Note that the large- N limit here is the weak convergence limit, which means that the matrices converge element by element. Furthermore, notice that non-zero matrix elements $\langle k | T_{(m,n)}^{(N)} | l \rangle$ in eq.(6) may change N by N . This causes a difference between the large- N limit of the product of the matrices and the product of the large- N limit of the matrices. On the other hand, if we adopt the PPWL operators in eq.(12), there appears no such problem since matrix elements $\langle k | \tilde{T}_{(m,n)}^{(N)} | l \rangle$ ($0 \leq k, l \leq N-1$) do not change in the large- N limit. Thus we can safely use the matrix model to regularize non-commutative field theory.

Finally, we shall comment on field theory with area-preserving diffeomorphism, which is usually argued as the large- N limit of matrix model [5, 6, 7]. At finite N , the basis of $N \times N$ Hermitian matrices are usually chosen by

$$\tilde{T}_{(m,n)}^{(N)}(\hat{x}, \hat{y}) = \frac{N}{2\pi} e^{2i\sqrt{\pi/N}(m\hat{x}+n\hat{y})}, \quad (23)$$

or

$$\tilde{M}_{(m,n)}^{(N)} = \frac{N}{2\pi} e^{i\frac{2\pi}{N}mn} \begin{pmatrix} 1 & & & \\ & e^{i\frac{4\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{4\pi}{N}(N-1)} \end{pmatrix}^m \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 1 & 0 & & 0 \end{pmatrix}^n, \quad (24)$$

instead of $T_{(m,n)}^{(N)}(\hat{x}, \hat{y})$ in eq.(1) or $M_{(m,n)}^{(N)}$ in eq.(5), respectively. In the large- N limit, their commutation relations are expected to become

$$[\tilde{T}_{(m_1,m_2)}^{(\infty)}(\hat{x}, \hat{y}), \tilde{T}_{(n_1,n_2)}^{(\infty)}(\hat{x}, \hat{y})] = -2i(m_1n_2 - m_2n_1) \tilde{T}_{(m_1+n_1, m_2+n_2)}^{(\infty)}(\hat{x}, \hat{y}), \quad (25)$$

or

$$[\tilde{M}_{(m_1,m_2)}^{(\infty)}, \tilde{M}_{(n_1,n_2)}^{(\infty)}] = -2i(m_1n_2 - m_2n_1) \tilde{M}_{(m_1+n_1, m_2+n_2)}^{(\infty)}, \quad (26)$$

which are the same algebra as the Poisson bracket of the plane waves on the two-dimensional torus. As we pointed out before, changing matrix elements when one enlarges the size of the matrix was the reason of breaking isomorphism in the large- N limit, thus we need a set of generators of the Poisson algebra represented on the Fock space to make PPWL operators, so that we can check the validity whether the field theory on torus with area-preserving diffeomorphism can be regularized by matrix model without breaking the isomorphism.

The following operators satisfy the same commutation relations as those of the Poisson bracket of the plane waves over two-dimensional torus [8],

$$\tilde{T}_{(m_1,m_2)}^{(P,\xi)} = \left\{ 1 - 2i\sqrt{\pi} \left(\xi m_1 \hat{a} + \frac{m_2}{\xi} \hat{a}^\dagger \right) \right\} \exp \left[2i\sqrt{\pi} \left(\xi m_1 \hat{a} + \frac{m_2}{\xi} \hat{a}^\dagger \right) \right], \quad (27)$$

where ξ is an arbitrary real constant. Then we would expect that a similar procedure holds. That is, once we take the N^2 numbers of the operators whose indices satisfy $-(N-1)/2 \leq m_1, m_2 \leq (N-1)/2$ and project out to the lowest N states, we would have the bases of the $N \times N$ Hermitian matrices as in the non-commutative torus case in eq.(12), and field theory with area-preserving diffeomorphism could also be regularized by matrix model. However, if we choose such a set of N^2 elements which include ⁴

$$\{\tilde{\mathcal{T}}_{(0,m)}^{(P,\xi)} : m = \pm 1, \pm 2, \dots, \pm \frac{N-1}{2}\}, \quad (28)$$

we find that these N elements are *not* independent. To prove this, we show that there exists a non-trivial set of $\{x_n\}$ for the equation,

$$\sum_n x_n \left(\sum_{k,l=0}^{N-1} |k\rangle \langle k| \tilde{\mathcal{T}}_{(0,n)}^{(P,\xi)} |l\rangle \langle l| \right) = 0, \quad (29)$$

where n runs over $-(N-1)/2, \dots, -1, 1, \dots, (N-1)/2$. Since the matrix elements of (28) are given by

$$\langle k | \tilde{\mathcal{T}}_{(0,n)}^{(P,\xi)} | l \rangle = \begin{cases} \sqrt{\frac{k!}{l!}} \frac{1}{(k-l-2)!} \left(-\frac{2\sqrt{\pi}}{\xi} n \right)^{k-l} & (k \geq l+2) \\ 0 & (k = l+1) \\ 1 & (k = l) \\ 0 & (k < l), \end{cases} \quad (30)$$

eq.(29) reduces to the following $N-1$ equations,

$$\begin{aligned} \sum_n x_n &= 0, \\ \sum_n \left(-\frac{2\sqrt{\pi}}{\xi} n \right)^2 x_n &= 0, \\ &\vdots \\ \sum_n \left(-\frac{2\sqrt{\pi}}{\xi} n \right)^{N-1} x_n &= 0. \end{aligned} \quad (31)$$

Thus, to prove the existence of non-trivial $\{x_n\}$, we have only to show

$$\det \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \left(\frac{2\sqrt{\pi}}{\xi} n_{\frac{N-1}{2}} \right)^2 & \dots & \left(\frac{2\sqrt{\pi}}{\xi} n_1 \right)^2 & \left(-\frac{2\sqrt{\pi}}{\xi} n_1 \right)^2 & \dots & \left(-\frac{2\sqrt{\pi}}{\xi} n_{\frac{N-1}{2}} \right)^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ \left(\frac{2\sqrt{\pi}}{\xi} n_{\frac{N-1}{2}} \right)^{N-1} & \dots & \left(\frac{2\sqrt{\pi}}{\xi} n_1 \right)^{N-1} & \left(-\frac{2\sqrt{\pi}}{\xi} n_1 \right)^{N-1} & \dots & \left(-\frac{2\sqrt{\pi}}{\xi} n_{\frac{N-1}{2}} \right)^{N-1} \end{pmatrix} = 0. \quad (32)$$

⁴We consider only odd N case.

And the above Vandermonde-like determinant can be straightforwardly shown to be zero by an elementary calculation. This implies that the similar projection does not work in this case and we should be very careful to regularize field theory with area-preserving diffeomorphism by matrix model [8]. This is consistent with the fact reported in [9, 10], where the N behavior of the matrix model and the lattice regularization of the corresponding field theory with area-preserving diffeomorphism have been numerically compared.

Acknowledgments:

The work of SU is supported in part by MEXT Grant-in-Aid for the Scientific Research #13135212.

References

- [1] N. Seiberg and E. Witten, “String Theory and Noncommutative Geometry,” JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].
- [2] J. Ambjørn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, “Lattice Gauge Fields and Discrete Noncommutative Yang-Mills Theory,” JHEP **0005**, 023 (2000) [arXiv:hep-th/0004147].
- [3] S. Minwalla, M. V. Raamsdonk and N. Seiberg, “Noncommutative Perturbative Dynamics,” JHEP **0002**, 020 (2000) [arXiv:hep-th/9912072].
M. Hayakawa, “Perturbative analysis on infrared aspects of noncommutative QED on R^4 ,” Phys. Lett. B **478**, 394 (2000) [arXiv:hep-th/9912094].
- [4] I. Bars and D. Minic, “Non-commutative geometry on a discrete periodic lattice and gauge theory,” Phys. Rev. D **62**, 105018 (2000) [arXiv:hep-th/9910091].
- [5] J. Hoppe, M.I.T. Ph. D. Thesis (1982).
- [6] B. de Wit, J. Hoppe and H. Nicolai, “On The Quantum Mechanics Of Supermembranes,” Nucl. Phys. B **305**, 545 (1988).
- [7] C. K. Zachos, “Hamiltonian Flows, $SU(\infty)$, $SO(\infty)$, $Usp(\infty)$, And Strings,” ANL-HEP-CP-89-55. *Presented at 18th Int. Conf. on Physics and Geometry, Tahoe City, CA, Jul 2-8, 1989.*
D. B. Fairlie, P. Fletcher and C. K. Zachos, “Trigonometric Structure Constants For New Infinite Algebras,” Phys. Lett. B **218**, 203 (1989).
D. B. Fairlie and C. K. Zachos, “Infinite Dimensional Algebras, Sine Brackets And $SU(\infty)$,” Phys. Lett. B **224**, 101 (1989).
D. B. Fairlie, P. Fletcher and C. K. Zachos, “Infinite Dimensional Algebras And A Trigonometric Basis For The Classical Lie Algebras,” J. Math. Phys. **31**, 1088 (1990).
- [8] N. Kitsunezaki and S. Uehara, “Representation of $SU(\infty)$ Algebra for Matrix Models,” JHEP **0210**, 024 (2002) [arXiv:hep-th/0206157].
- [9] N. Kitsunezaki and S. Uehara, “Large N structure of IIB matrix model,” Prog. Theor. Phys. **105**, 515 (2001) [arXiv:hep-th/0010038];

N. Kitsunezaki and S. Uehara, “Large N behaviors of the IIB matrix model and the regularized Schild models,” *JHEP* **0110**, 033 (2001) [arXiv:hep-th/0108181].

- [10] K. N. Anagnostopoulos, J. Nishimura and P. Olesen, “Noncommutative string world-sheets from matrix models,” *JHEP* **0104**, 024 (2001) [arXiv:hep-th/0012061].